Limiting of Signals in Random Noise

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Abstract—**The effect of ideal bandpass limiting on a signal lying in narrowband Gaussian noise is analyzed. General analytic expressions for the limiter output components are derived using an integral representation for the limiter characteristic. This method allows retention of the phases of all the signals and the intermodulation products at the limiter output, which are destroyed in the characteristic-function method generally used in limiter studies. Expressions for the desired signals and intermodulation product amplitudes are obtained for the case when the limiter input consists of three angle-modulated sinusoids and noise. The analysis is extended to *n* modulated sinusoids plus noise, and approximate expressions for the signal, intermodulation product, and noise terms are derived. Numerical results are presented for the signal suppression and the limiter output signal amplitudes for the case of three input signals, two of equal amplitude.**

Introduction

THE calculation of the output of a limiting device when its input consists of a sum of several signals has been the subject of a great deal of theoretical analysis and has resulted in a number of widely quoted publications. Essentially, two methods of approach have been used to analyze the problem. The first approach, commonly known as the characteristic-function method of Rice, involves computing the autocorrelation function of the limiter output and then taking the Fourier transform to obtain the power-density spectrum. Although a general expression for the limiter output autocorrelation function can be derived, its computation becomes extremely involved when modulation of the input signals is considered. The difficulty lies in determining the characteristic function of the signals with arbitrary modulation. However, if the signals are statistically independent angle-modulated sinusoids, and only the average power or the magnitude of the signal and cross-product terms at the limiter output is of interest, the modulation of the input signals can be ignored, which considerably simplifies the analysis. Davenport [1] was first to use this approach to investigate the effect of hard limiting a single sinusoidal signal and narrowband Gaussian noise. Jones [2] used the same method to analyze the case of two sinusoids plus noise. More recently, Shaft [3] and Gyi [4] independently extended the analyses to include *n* sinusoids. The magnitude of any signal or cross product is given by an untabulated infinite integral that has been numerically evaluated for a number of cases of interest.

In many practical applications, the phases of the signal and cross-product terms are also of interest. For example, in an FM or PM system, the spreading of the power spectral density of the signal and cross-product terms will be a function of their phase modulation, even though the average power in any one output component is independent of its phase. Thus, modulation of the input signals must be considered in the analysis, in order to retain the phase of the signal and cross-product terms at the output of the limiter.

The second method, which can be referred to as the Fourier-expansion approach, is carried out entirely in the time domain, in contrast to the autocorrelation approach, which treats the problem in the frequency domain. Analysis in the time domain does not lose the phase of the signal and cross-product terms. It also allows removal of the assumption of statistical independence between the input signals. This approach was used by Granlund [5] and later by Baghdady [6] to investigate the noiseless case of two sinusoids passed through an ideal bandpass limiter. More recently, Sollfrey [7] used the same approach to analyze the effect of hard limiting on a sum of three or four sinusoidal signals without noise. Closed-form analytic expressions for the amplitude of the desired signal terms were obtained for three input signals, two of equal amplitude, and for four signals, having two pairs of equal amplitude. The Fourier expansion method, however, has the drawback that it is difficult to consider the effect of noise present at the input to the limiter. The above references, therefore, do not consider noise and are directed primarily to the calculation of signal amplitude at the output of the limiter.

The purpose of this paper is to extend the Fourier-expansion method to include random noise in the analysis, and to derive a general analytical expression for the output of a limiting device. The approach is similar to that used by Reed [9] for calculating the amplitudes of two signals in noise. Closed-form expressions for the desired signal and cross-product amplitudes are presented for the case of three modulated sinusoids and noise. The analysis is then extended to *n* modulated sinusoids plus noise, and approximate expressions for the signal, cross-product, and noise terms are derived.

Calculation of the Limiter Output

The specific model for the bandpass limiter to be considered is shown in Fig. 1. The input to the limiter

$$x(t) = s(t) + n(t) \tag{1}$$

consists of the signal $s(t)$ and a band of zero-mean stationary Gaussian noise $n(t)$. It is assumed that the bandpass filter preceding the limiter is wide enough to pass the signal with negligible distortion and limits the input noise to a narrow bandwidth that is small compared to the center frequency of the filter. The limiter is followed by another bandpass filter that confines the output spectrum essentially only to the fundamental band of the signal.

It is assumed that the limiter has a hard-limiting character-

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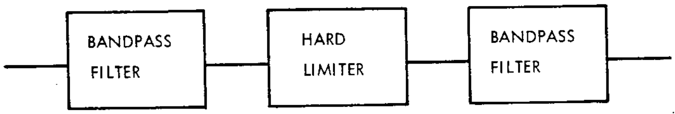

Fig. 1. Model for investigation of ideal symmetric limiting.

istic that limits its output to either ± 1. Thus, if the limiter input is $x(t)$, the output $y(t)$ may be expressed in analytical form [7] as

$$y(t) = \frac{2}{\pi} \int_0^\infty \sin[vx(t)] \frac{dv}{v}. \tag{2}$$

When the expression (1) is inserted into this integral, the sine of a sum may be expanded into the sum of two products of sine and cosine. Thus,

$$y(t) = \frac{2}{\pi} \int_0^\infty \sin[vs(t)] \cos[vn(t)] \frac{dv}{v} + \frac{2}{\pi} \int_0^\infty \cos[vs(t)] \sin[vn(t)] \frac{dv}{v}. \tag{3}$$

The narrowband Gaussian noise at the limiter input may be expressed as

$$n(t) = r \cos(\omega_0 t + \phi), \tag{4}$$

where the envelope r and the phase ϕ are slowly time-varying random variables having Rayleigh and uniform distributions, respectively.

Substituting the above expression for the noise in (3), and using the general relationships

$$\sin(z \cos p) = 2 \sum_{m=0}^\infty (-1)^m J_{2m+1}(z) \cos(2m+1)p$$

$$\cos(z \cos p) = J_0(z) + 2 \sum_{m=1}^\infty (-1)^m J_{2m}(z) \cos 2mp, \tag{5}$$

the following expression for the limiter output is obtained:

$$\begin{aligned} y(t) = {} & \frac{2}{\pi} \int_0^\infty \sin[vs(t)] \cdot J_0(vr) \frac{dv}{v} \\ & + \frac{4}{\pi} \sum_{m=1}^\infty (-1)^m \int_0^\infty \sin[vs(t)] \\ & \cdot J_{2m}(vr) \frac{dv}{v} \cos 2m(\omega_0 t + \phi) \\ & + \frac{4}{\pi} \sum_{m=0}^\infty (-1)^m \int_0^\infty \cos[vs(t)] \\ & \cdot J_{2m+1}(vr) \frac{dv}{v} \cos(2m+1)(\omega_0 t + \phi). \end{aligned} \tag{6}$$

All the terms in the above equation are random functions due to the presence of the noise envelope r and phase ϕ. However, only the first integral will yield an average output, while the two other integrals will not contribute since all the terms contain the random phase of the noise.

The average limiter output at any arbitrary time t is obtained by averaging over all possible noise amplitudes, as follows:

$$z(t) = E[y(t)] = \frac{2}{\pi} \int_0^\infty \sin[vs(t)] \cdot E[J_0(vr)] \frac{dv}{v}. \tag{7}$$

Since the noise envelope has a Rayleigh distribution

$$p(r) = \frac{r}{\sigma^2} \exp\left[-\frac{r^2}{2\sigma^2}\right], \tag{8}$$

where σ^2 is the total noise power at the limiter input, the expected value of $J_0(vr)$ is given by [8]

$$E[J_0(vr)] = \int_0^\infty J_0(vr) p(r)\, dr = \exp\left[-\frac{v^2\sigma^2}{2}\right]. \tag{9}$$

Substitution of (9) in (7) yields

$$z(t) = \frac{2}{\pi} \int_0^\infty \sin[vs(t)] \cdot \exp\left[-\frac{v^2\sigma^2}{2}\right] \frac{dv}{v}. \tag{10}$$

This is identical to the expression obtained by Reed [9] for calculating the amplitudes of two signals in noise. The above integral may also be written in terms of the error function [10] as

$$z(t) = \operatorname{erf}\left(\frac{s(t)}{\sqrt{2}\,\sigma}\right). \tag{11}$$

The presence of noise at the input of a hard limiter thus tends to make the limiting soft, due to the gradual saturation characteristics of the error function. It may be seen, as also observed by Jones [2], that had a smooth limiter having an error function amplitude characteristic been used, the only change in (10) and (11) would have been to replace σ^2 by $\sigma^2 + \gamma^2$. The quantity γ determines the slope of the limiter characteristic, i.e., how fast saturation is approached in the case of soft limiting. The effect of soft limiting on the average output is therefore mathematically identical to adding noise to a hard limiter, provided that an error-function representation can be used for the limiter characteristic.

The components that collectively make up the limiter output noise spectrum are the second and third integrals of (6):

$$\begin{aligned} \eta(t) = {} & \frac{4}{\pi} \sum_{m=1}^\infty (-1)^m \int_0^\infty \sin[vs(t)] \\ & \cdot J_{2m}(vr) \frac{dv}{v} \cos 2m(\omega_0 t + \phi) \\ & + \frac{4}{\pi} \sum_{m=0}^\infty (-1)^m \int_0^\infty \cos[vs(t)] \\ & \cdot J_{2m+1}(vr) \frac{dv}{v} \cos(2m+1)(\omega_0 t + \phi). \end{aligned} \tag{12}$$

The expressions (10) and (12) are the two fundamental equations for the analysis of the limiter in the time domain. In Davenport's notation [1], (10) represents the output signal and intermodulation components resulting from the interaction of the signal with itself ($s \times s$ terms). Similarly, (12) represents output noise components that are due to

the interaction of the input noise with itself ($n \times n$ terms) and with the input signal ($s \times n$ terms). A separation of the noise output into $n \times n$ terms and $s \times n$ terms is not possible, without the knowledge of the signal waveform $s(t)$.

If the characteristic-function method had been used to calculate the limiter output, the phases of all the components at the limiter output would have been lost, and a separation of the output components as in (10) and (12) would have been possible only in terms of the autocorrelation function of the individual components.

Limiting of Three Signals and Noise

The effect of ideal bandpass limiting upon one and two sinusoids lying in narrowband Gaussian noise has been treated by several authors. This section presents analytic results for the case of three angle-modulated sinusoidal input signals plus noise. Let the signals at the input to the limiter be

$$\begin{aligned} s(t) &= a\cos[\omega_1 t + \phi_1(t)] + b\cos[\omega_2 t + \phi_2(t)] \\ &\quad + c\cos[\omega_3 t + \phi_3(t)] \\ &= a\cos r + b\cos s + c\cos t, \end{aligned} \tag{13}$$

where $r = \omega_1 t + \phi_1(t)$, and similarly for s and t.

When (13) is inserted into (10), the sine of a sum may be transformed by simple trigonometry into the sum of four products of sines. Thus

$$\begin{aligned} z(t) = \frac{2}{\pi}\int_0^\infty \exp\left[-\frac{v^2\sigma^2}{2}\right]\frac{dv}{v} & \\ \cdot [&\sin(va\cos r)\cos(vb\cos s)\cos(vc\cos t) \\ &+ \sin(vb\cos s)\cos(vc\cos t)\cos(va\cos r) \\ &+ \sin(vc\cos t)\cos(va\cos r)\cos(vb\cos s) \\ &- \sin(va\cos r)\sin(vb\cos s)\sin(vc\cos t)]. \end{aligned} \tag{14}$$

Using (5), the sines and cosines may be expanded as Fourier series in r, s, t, whose coefficients are Bessel functions. The output of the limiter prior to filtering may, therefore, be written as

$$\begin{aligned} z(t) = 4\sum_{n=0}^\infty\sum_{m=0}^\infty\sum_{q=0}^\infty & (-1)^{n+m+q}\varepsilon_m\cdot\varepsilon_q \\ &\cdot h_{2n+1,2m,2q}\cos(2n+1)r\cos(2ms)\cos(2qt) \\ &+ 4\sum_{n=0}^\infty\sum_{m=0}^\infty\sum_{q=0}^\infty(-1)^{n+m+q}\varepsilon_m\cdot\varepsilon_q \\ &\cdot h_{2q,2n+1,2m}\cos(2qr)\cos(2n+1)s\cos(2mt) \\ &+ 4\sum_{n=0}^\infty\sum_{m=0}^\infty\sum_{q=0}^\infty(-1)^{n+m+q}\varepsilon_m\cdot\varepsilon_q \\ &\cdot h_{2m,2q,2n+1}\cos(2mr)\cos(2qs)\cos(2n+1)t \\ &- 4\sum_{n=0}^\infty\sum_{m=0}^\infty\sum_{q=0}^\infty(-1)^{n+m+q}h_{2n+1,2m+1,2q+1} \\ &\cdot\cos(2n+1)r\cos(2m+1)s\cos(2q+1)t, \end{aligned} \tag{15}$$

where $h_{\mu\eta\xi}$ represents the integral

$$h_{\mu\eta\xi} = \frac{4}{\pi}\int_0^\infty \frac{J_\mu(va)J_\eta(vb)J_\xi(vc)}{v}\exp\left[-\frac{v^2\sigma^2}{2}\right]dv. \tag{16}$$

The output of the limiter can be arranged in terms of signal and cross-product components by representing each of the products of the three cosines in (15) as a sum of four cosines using the trigonometric identity:

$$\begin{aligned} \cos\alpha\cdot\cos\beta\cdot\cos\gamma = \tfrac{1}{4}[&\cos(\alpha+\beta+\gamma) \\ &+ \cos(\alpha+\beta-\gamma) \\ &+ \cos(\alpha-\beta+\gamma) \\ &+ \cos(\alpha-\beta-\gamma)]. \end{aligned} \tag{17}$$

The amplitude of any desired signal or cross-product component may be obtained from (16). The signal and the strongest cross-product components are obtained by setting $n = m = q = 0$ in (15):

$$\begin{aligned} z(t) \approx h_{100}\cos r &+ h_{010}\cos s + h_{001}\cos t \\ &- h_{111}[\cos(r+s+t) + \cos(r+s-t) \\ &+ \cos(r-s+t) + \cos(r-s-t)]. \end{aligned} \tag{18}$$

To evaluate the coefficient $h_{\mu\eta\xi}$, it is convenient to begin with the power-series expansion of the product of two Bessel functions [11]:

$$\begin{aligned} J_\eta(vb)\cdot J_\xi(vc) = \frac{1}{\eta!}(\tfrac{1}{2}vb)^\eta(\tfrac{1}{2}vc)^\xi\sum_{p=0}^\infty\frac{(-1)^p(\tfrac{1}{2}vc)^{2p}}{p!\,(p+\xi)!} & \\ \cdot {}_2F_1\left(-p,-p-\xi,\eta+1,\frac{b^2}{c^2}\right), & \end{aligned} \tag{19}$$

where ${}_2F_1(\ )$ is the Gaussian hypergeometric function.

Substitution of the above in (16) yields

$$\begin{aligned} h_{\mu\eta\xi} = \frac{4}{\pi(\eta!)}\left(\frac{b}{2}\right)^\eta\left(\frac{c}{2}\right)^\xi\sum_{p=0}^\infty\frac{(-1)^p}{p!\,(p+\xi)!} & \\ \cdot\left(\frac{c}{2}\right)^{2p}{}_2F_1\left(-p,-p-\xi,\eta+1,\frac{b^2}{c^2}\right) & \\ \cdot\int_0^\infty J_\mu(va)\cdot v^{\eta+\xi+2p-1}\cdot\exp\left[-\frac{v^2\sigma^2}{2}\right]dv. & \end{aligned} \tag{20}$$

The solution of the above integral, which is attributed to Weber and Sonine [8], is given in terms of confluent hypergeometric function. The coefficient $h_{\mu\eta\xi}$ may be written explicitly as

$$\begin{aligned} h_{\mu\eta\xi} = \frac{2}{\pi(\mu!)(\eta!)}\left(\frac{a}{\sqrt{2}\,\sigma}\right)^\mu\left(\frac{b}{\sqrt{2}\,\sigma}\right)^\eta\left(\frac{c}{\sqrt{2}\,\sigma}\right)^\xi & \\ \cdot\sum_{p=0}^\infty\frac{(-1)^p}{p!\,(p+\xi)!}\Gamma\left(p+\frac{\mu+\eta+\xi}{2}\right) & \\ \cdot\left(\frac{c}{\sqrt{2}\,\sigma}\right)^{2p}\cdot{}_2F_1\left(-p,-p-\xi,\eta+1,\frac{b^2}{c^2}\right) & \\ \cdot{}_1F_1\left(p+\frac{\mu+\eta+\xi}{2},\mu+1,-\frac{a^2}{2\sigma^2}\right), & \\ \mu+\eta+\xi \text{ odd}. & \end{aligned} \tag{21}$$

This form is useful for numerical computations because the Gaussian hypergeometric function terminates after the $(p+1)$th term due to the parameter $-p$. The series representation for $h_{\mu\eta\xi}$ is convergent, unless one or more of the input signals rise substantially above the input noise level, or the noise is weak ($\sigma^2 \approx 0$).[1] In this case, the convergence of (21) is dependent on the relative amplitude of the three signals. In the general case of three signals of arbitrary amplitudes in weak noise, a different representation for the h coefficients is required. This will be treated later in this section.

In the special case of two signals having equal amplitude, $(b = c)$, the expressions for $h_{\mu\eta\xi}$ may be simplified by using the identity

$$ {}_2F_1(\alpha,\beta,\gamma,1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}, \qquad \gamma \neq 0, -1, -2, \cdots, \quad \operatorname{Re}(\gamma-\alpha-\beta) > 0 \tag{22} $$

for the Gaussian hypergeometric function. If the input noise can be neglected, the expressions may be further simplified by using the asymptotic expansion for the confluent hypergeometric function:

$$ {}_1F_1(\alpha, \beta, -Z) \approx \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)Z^{\alpha}}\left[1 + \frac{\alpha(\alpha-\beta+1)}{Z} + \cdots\right], \qquad Z \to \infty. \tag{23} $$

If only the first term in the series expansion is taken, the expressions for the amplitude of the desired signals are then identical with the results obtained by Sollfrey [7]. Furthermore, we also obtain expressions for the amplitude of the cross-product terms, which are not given by Sollfrey. The expression for $h_{\mu\eta\xi}$ is convergent, however, only if a is greater than $2b$. For the case where $2b$ is greater than a and noise is weak, $h_{\mu\eta\xi}$ may be evaluated by the method used by Sollfrey for the noiseless case. It involves replacing the product of the two Bessel functions of the same argument in (16) by a contour integral representation

$$ J_\eta(vb)\cdot J_\xi(vb) = \frac{1}{2\pi i}\int_L \frac{\Gamma(-s)\cdot\Gamma(\eta+\xi+2s+1)}{\Gamma(\eta+s+1)\cdot\Gamma(\xi+s+1)\cdot\Gamma(\eta+\xi+s+1)} \cdot \left(\frac{vb}{2}\right)^{\eta+\xi+2s} ds \tag{24} $$

and then solving the double integral by changing the order of integration, i.e., by evaluating the real integral over v first. The real integral in this case is that of Weber and Sonine [8], analogous to that of (20), and its solution is obtained in terms of the confluent hypergeometric function. The result is

$$ h_{\mu\eta\xi} = \frac{2}{\pi(\mu!)}\left(\frac{a}{\sqrt{2}\,\sigma}\right)^{\mu}\left(\frac{b}{\sqrt{2}\,\sigma}\right)^{\eta+\xi}\cdot\frac{1}{2\pi i} \cdot\int_L \frac{\Gamma(-s)\cdot\Gamma(\eta+\xi+2s+1)\cdot\Gamma[s+(\mu+\eta+\xi)/2]}{\Gamma(\eta+s+1)\cdot\Gamma(\xi+s+1)\cdot\Gamma(\eta+\xi+s+1)} \cdot\left(\frac{b}{\sqrt{2}\,\sigma}\right)^{2s}\cdot{}_1F_1\left(s+\frac{\mu+\eta+\xi}{2}, \mu+1, -\frac{a^2}{2\sigma^2}\right) ds. \tag{25} $$

For large values of signal-to-noise ratio $a^2/2\sigma^2$, the asymptotic expansion of (23) for the confluent hypergeometric function can be used for evaluating the integral. Unfortunately, the integral converges only for the first two terms of the series expansion. Thus, the expression for $h_{\mu\eta\xi}$ will be valid only for large values of $a^2/2\sigma^2$. Using the method of residues for solving the integral, the following expressions for the amplitudes of the desired signal components are obtained:

$$ \begin{aligned} h_{100} = {} & \frac{2}{\pi^{7/2}}\sum_{n=0}^{\infty}\frac{[\Gamma(n+\frac{1}{2})]^3}{(n!)^2(n+1)!}\left(\frac{a}{2b}\right)^{2n+1} \cdot\left[\frac{1}{n+1} + 3\{\psi(n+1) - \psi(n+\tfrac{1}{2})\} + 2\log\frac{2b}{a}\right] \\ & + \frac{2}{\pi^{7/2}}\left(\frac{\sigma}{\sqrt{2}\,b}\right)^2\sum_{n=0}^{\infty}\frac{[\Gamma(n+\frac{3}{2})]^3}{n!\,[(n+1)!]^2}\left(\frac{a}{2b}\right)^{2n+1} \cdot\left[\frac{2}{n+1} - \frac{6}{2n+1} + 3\{\psi(n+1) - \psi(n+\tfrac{1}{2})\} + 2\log\frac{2b}{a}\right] - \frac{2}{\pi^2}\left(\frac{\sigma^2}{ab}\right), \qquad \frac{2b}{a} > 1 \end{aligned} \tag{26} $$

$$ \begin{aligned} h_{010} = {} & \frac{8}{\pi^2} - \frac{2}{\pi^{7/2}}\sum_{n=0}^{\infty}\frac{[\Gamma(n+\frac{1}{2})]^2\cdot\Gamma(n+\frac{3}{2})}{n!\,[(n+1)!]^2}\left(\frac{a}{2b}\right)^{2n+2} \cdot\left[\frac{2n}{(n+1)(2n+1)} + 3\{\psi(n+1) - \psi(n+\tfrac{1}{2})\} + 2\log\frac{2b}{a}\right] \\ & - \frac{2}{\pi^{7/2}}\left(\frac{\sigma}{\sqrt{2}\,b}\right)^2\sum_{n=0}^{\infty}\frac{[\Gamma(n+\frac{1}{2})]^2\cdot\Gamma(n+\frac{3}{2})}{(n!)^3}\left(\frac{a}{2b}\right)^{2n} \cdot\left[-\frac{2}{2n+1} + 3\{\psi(n+1) - \psi(n+\tfrac{1}{2})\} + 2\log\frac{2b}{a}\right], \qquad \frac{2b}{a} > 1. \end{aligned} \tag{27} $$

Here, $\psi(\;)$ denotes the logarithmic derivative of the gamma function. Similar expressions can also be derived for the desired cross-product components; the calculation of the residues, however, is quite tedious. In the absence of input noise ($\sigma^2 = 0$), the expressions are identical with the results of Sollfrey [7].

[1] The series in (21) is convergent for all parameter values except $\sigma^2 = 0$.

To evaluate the coefficients $h_{\mu\eta\xi}$ for arbitrary signal amplitudes and weak noise, it is convenient to use Bateman's expansion [12] for the product of two Bessel functions:

$$J_\eta(vb) \cdot J_\xi(vc) = 2b^\eta \cdot c^\xi[2(b^2 + c^2)]^{-1/2(\eta+\xi+1)} \cdot \sum_{n=0}^{\infty} f(\eta,\xi,n,b,c) \cdot \frac{J_{\eta+\xi+2n+1}(v\sqrt{2(b^2 + c^2)})}{v}, \tag{28}$$

where

$$f(\eta,\xi,n,b,c) = \frac{(-1)^n \cdot (\eta + \xi + 2n + 1) \cdot \Gamma(\eta + \xi + n + 1) \cdot \Gamma(\xi + n + 1)}{n! \cdot \Gamma(\eta + n + 1) \cdot \Gamma(\xi + 1) \cdot \Gamma(\xi + 1)} \cdot {}_2F_1(-n, \eta + \xi + n + 1, \xi + 1, \tfrac{1}{2}) \cdot {}_2F_1\left(-n, \eta + \xi + n + 1, \xi + 1, \frac{c^2}{b^2 + c^2}\right). \tag{29}$$

Both the hypergeometric functions terminate after the $(n + 1)$th term due to the parameter $-n$.

Substitution of (28) in (16) yields

$$h_{\mu\eta\xi} = \frac{8}{\pi} b^\eta \cdot c^\xi \cdot [2(b^2 + c^2)]^{-1/2(\eta+\xi+1)} \sum_{n=0}^{\infty} f(\eta,\xi,n,b,c) \cdot \int_0^\infty \frac{J_\mu(va) \cdot J_{\eta+\xi+2n+1}(v\sqrt{2(b^2 + c^2)})}{v^2} \cdot \exp\left[-\frac{v^2\sigma^2}{2}\right] dv. \tag{30}$$

The integral can be easily evaluated by replacing the product of the Bessel functions by the series representation of (19). In the absence of noise, the integral represents the discontinuous integral of Weber and Schafheitlin [13]. Its solution depends on the relative magnitude of the arguments of the two Bessel functions. In the following, only the solution for $a < \sqrt{2(b^2 + c^2)}$ will be considered; the case $a \geq \sqrt{2(b^2 + c^2)}$ can be treated similarly. Using the appropriate solution of the integral [13], the expression for $h_{\mu\eta\xi}$ may be written as

$$h_{\mu\eta\xi} = \frac{2}{\pi} a^\mu \cdot b^\eta \cdot c^\xi[2(b^2 + c^2)]^{-1/2(\mu+\eta+\xi)} \sum_{n=0}^{\infty} f(\eta,\xi,n,b,c) \cdot \frac{\Gamma[n + \frac{1}{2}(\mu + \eta + \xi)]}{\mu! \cdot \Gamma[n + 2 + \frac{1}{2}(\eta + \xi - \mu)]} \cdot {}_2F_1\left(n + \tfrac{1}{2}(\mu + \eta + \xi), -n - 1 + \tfrac{1}{2}(\mu - \eta - \xi), \mu + 1, \frac{a^2}{2(b^2 + c^2)}\right), \qquad \sqrt{2(b^2 + c^2)} > a. \tag{31}$$

This expression is convergent and yields the amplitudes of the signals and the cross products in the absence of noise. In most cases, it would be possible to simplify the expression for any h coefficient by expressing the hypergeometric function containing the factor $\frac{1}{2}$ in terms of gamma functions. For example, the expression for h_{100},

$$h_{100} = \frac{2}{\pi}\sqrt{\frac{2a^2}{(b^2 + c^2)}} \sum_{n=0}^{\infty} (-1)^n {}_2F_1(-n, n + 1, 1, \tfrac{1}{2}) \cdot {}_2F_1\left(-n, n + 1, 1, \frac{c^2}{b^2 + c^2}\right) \cdot {}_2F_1\left(n + \tfrac{1}{2}, -n - \tfrac{1}{2}, 2, \frac{a^2}{2(b^2 + c^2)}\right) \tag{32}$$

may be simplified by using the identity

$$ {}_2F_1(-n, n + 1, 1, \tfrac{1}{2}) = \sqrt{\pi}\left[\Gamma\left(\frac{1}{2} - \frac{n}{2}\right) \cdot \Gamma\left(1 + \frac{n}{2}\right)\right]^{-1}. \tag{33}$$

It is seen that the series is zero for all odd integer values of n. In the special case of two signals having equal amplitudes, the second hypergeometric function can also be replaced by (33), yielding the simple expression

$$h_{100} = \frac{2}{\pi^2}\left(\frac{a}{b}\right) \sum_{n=0}^{\infty} \frac{[\Gamma(n + \frac{1}{2})]^2}{(n!)^2} \cdot {}_2F_1\left(2n + \tfrac{1}{2}, -2n - \tfrac{1}{2}, 2, \frac{a^2}{4b^2}\right), \qquad \frac{2b}{a} > 1. \tag{34}$$

Similarly, for h_{010} the following expression is obtained

$$h_{010} = h_{001} = \frac{8}{\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n \cdot \Gamma(n + \frac{1}{2})}{(n + 1) \cdot \Gamma(n + \frac{5}{2})} \cdot {}_2F_1\left(n + \tfrac{1}{2}, -n - \tfrac{3}{2}, 1, \frac{a^2}{4b^2}\right) \cdot \left\{\frac{\Gamma\left(\frac{3}{2} + \frac{n}{2}\right)\cos n\pi/2}{\Gamma\left(1 + \frac{n}{2}\right)} - \frac{\Gamma\left(1 + \frac{n}{2}\right)\sin n\pi/2}{\Gamma\left(\frac{1}{2} + \frac{n}{2}\right)}\right\}^2, \qquad \frac{2b}{a} > 1 \tag{35}$$

by using the expression in the brackets to replace $\frac{1}{2}\sqrt{\pi} \cdot (n + 1) \cdot {}_2F_1(-n, n + 2, 1, \frac{1}{2})$. Numerical evaluation of (34) and (35) yield results identical to those obtained from (26) and (27) in the absence of noise; however, it has not been possible to convert one form into the other. Similar expressions for the cross products can be readily derived. This approach has the advantage over Sollfrey's method in that it directly provides the expressions for the signals and all the cross products. Thus, there is no need to

calculate the residues for each h coefficient, as required in the evaluation of (25).

Numerical Results

The results for limiting of three signals in noise are shown in Figs. 2–6. In all cases, the amplitudes of two signals were assumed to be equal ($b = c$), in order to reduce the number of variables. The following expressions were used for numerical computations:

$$h_{100} = \frac{2}{\pi}\left(\frac{a}{\sqrt{2}\,\sigma}\right)\sum_{p=0}^{\infty}\frac{(-1)^p\cdot(2p)!\cdot\Gamma(p+\frac{1}{2})}{(p!)^4}\left(\frac{b}{\sqrt{2}\,\sigma}\right)^{2p} \cdot {}_1F_1\left(p+\tfrac{1}{2}, 2, -\frac{a^2}{2\sigma^2}\right) \quad (36)$$

$$h_{010} = \frac{2}{\pi}\left(\frac{b}{\sqrt{2}\,\sigma}\right)\sum_{p=0}^{\infty}\frac{(-1)^p\cdot(2p+1)!\cdot\Gamma(p+\frac{1}{2})}{(p!)^2\cdot[(p+1)!]^2} \cdot\left(\frac{b}{\sqrt{2}\,\sigma}\right)^{2p}\cdot {}_1F_1\left(p+\tfrac{1}{2}, 1, -\frac{a^2}{2\sigma^2}\right). \quad (37)$$

These are obtained from (21) by replacing the Gaussian hypergeometric function with the identity of (22). When the noise is weak, computation of these expressions breaks down and it is necessary to employ the asymptotic expansion of (23) for the confluent hypergeometric function. The resulting expressions, however, are convergent only if a is greater than $2b$. For the case a less than $2b$ (26) and (27) are applicable.

The curves in Figs. 2 and 3 show the amplitudes h_{100} and h_{010} plotted against the limiter input power ratio of double to single component with the limiter input signal-to-noise ratio of the single component as parameter. For b/a small, that is, strong single and weak double component, the results approach the one-signal-in-noise case discussed by Davenport [1]. In the absence of noise, the single component tends to $4/\pi$ and gets essentially all of the limiter output power. As the signal-to-noise ratio decreases, the noise uses up some of the output power, so that the proportion used by the strong single component decreases. As the ratio b/a increases, the proportion of the output power used by the double component also increases, while that of the single component decreases. When the double component is strong, that is b/a is large, the results approach the two-signal case discussed by Jones [2]. The double component tends to $8/\pi^2$, when there is no noise. Each of the two strong signals receives -3.9 dB of the output power, so that 0.9 dB remains for the weak single component and cross products. As the signal-to-noise ratio decreases, the share of the output power consumed by noise increases, and consequently the power in the double component decreases. The plot in Fig. 4 shows the result of limiting three signals of equal amplitude in noise.

Signal suppression is another phenomena of interest. It is defined as the ratio of the weak-to-strong component ratio at the limiter output to the corresponding quantity

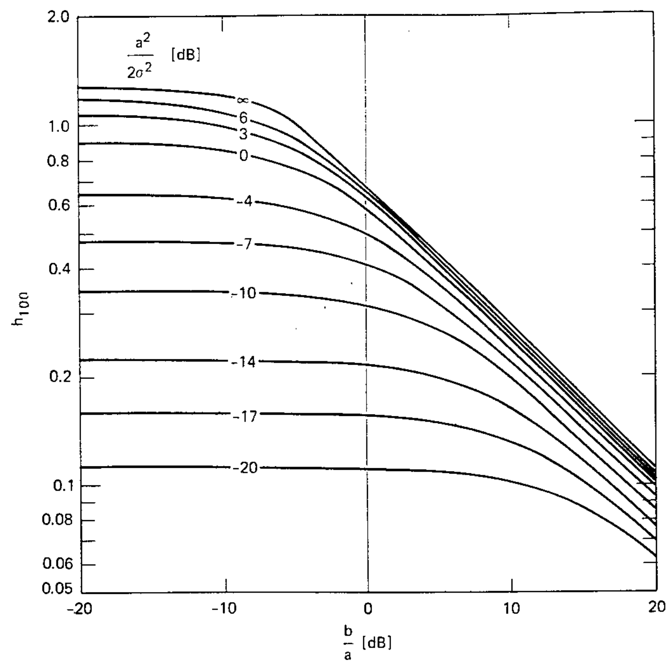

Fig. 2. Amplitude of the limiter output single component—three inputs.

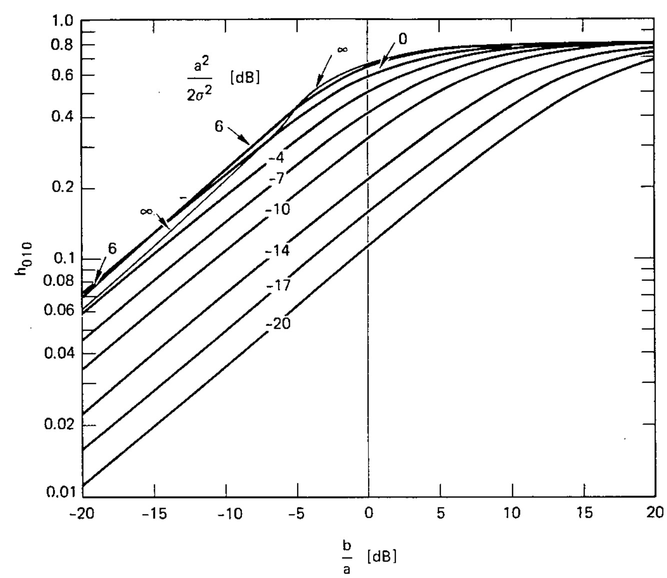

Fig. 3. Amplitude of the limiter output double component—three inputs.

at the input. In decibels the suppression ratio is given by

$$L = 20\log\frac{h_{100}}{h_{010}} - 20\log\frac{a}{b}, \qquad \frac{a}{b} > 1$$

$$= 20\log\frac{h_{010}}{h_{100}} - 20\log\frac{b}{a}, \qquad \frac{a}{b} < 1. \quad (38)$$

Positive decibel values of L correspond to reduction of the weak component. The first case corresponds to a strong single component and weak double components, the second to weak single and strong double. In Figs. 5 and 6,

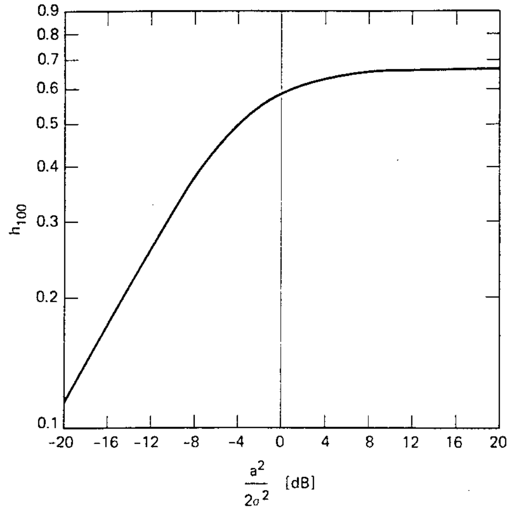

Fig. 4. Amplitude of the limiter output signal components, three inputs equal in amplitude.

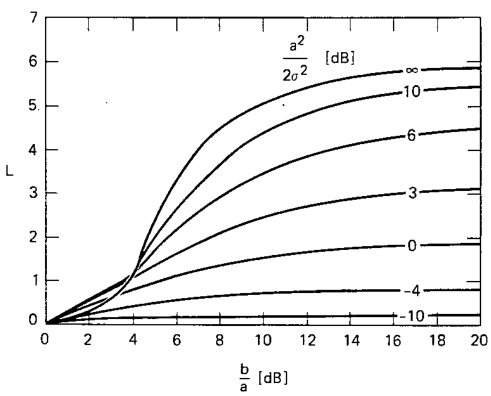

Fig. 5. Three-signal suppression—one strong, two weak signals.

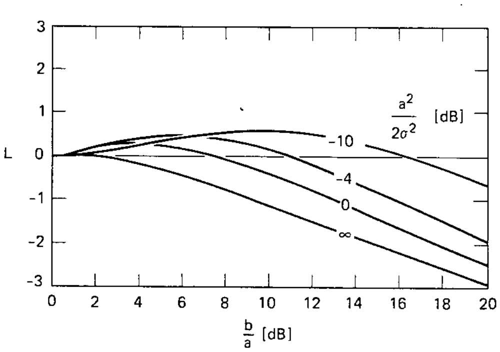

Fig. 6. Three-signal suppression—one weak, two strong signals.

the suppression ratio L is plotted against the input ratio b/a in absolute decibels.

For the one-strong-two-weak case shown in Fig. 5, the maximum suppression of the weak double component occurs at large input signal-to-noise ratios, and it is seen to be 6 dB when there is no noise. On the other hand, when the noise is strong, the limiter acts essentially as a linear device, in the sense that almost no suppression of the weak double component takes place. This behavior is in complete agreement with the suppression of a weak signal by a strong signal, as influenced by the noise level, for the two-signal case discussed by Jones [2]. In the case of strong double and weak single components, shown in Fig. 6, hard-limiting enhances the weaker component with respect to the stronger double components at large values of b/a. Thus, under these circumstances, the limiter displays "negative suppression." Maximum enhancement of the weak signal occurs in the absence of noise.

This interesting phenomenon was originally discovered by Jones [2] for the case where the weak signal was noise rather than another sinusoid. In the case of three signals, the effect arises from beats between the two equal strong components, which permit the weak component to slip through. It also appears in the results of Shaft [3].

The asymptotic curves in Figs. 2–6 describing the noiseless case were first obtained by Sollfrey [7].

Limiting of n Signals and Noise

The method used to analyze the case of three input signals may be extended to the general case of n angle-modulated sinusoids. The signal and intermodulation products at the limiter output are obtained by inserting the expression for the limiter input signals

$$s(t) = \sum_{i=1}^{n} a_i \cos\left[\omega_i t + \phi_i(t)\right] \tag{39}$$

into (10) and using the general relationship

$$\sin\left(\sum_{i=1}^{n} \beta_i \cos \alpha_i\right) = \sum_{p_1=-\infty}^{\infty} \cdots \sum_{p_n=-\infty}^{\infty} \left[\prod_{i=1}^{n} J_{p_i}(\beta_i) \sin\left(\sum_{i=1}^{n} p_i\left(\alpha_i + \frac{\pi}{2}\right)\right)\right], \tag{40}$$

where the symbol $\prod$ indicates that all n of the J_{p_i} coefficients are multiplied together. Thus, the limiter output, prior to bandpass filtering, is given by

$$z(t) = \tfrac{1}{2} \sum_{p_1=-\infty}^{\infty} \cdots \sum_{p_n=-\infty}^{\infty} h_{p_1 p_2 \cdots p_n} \cdot \sin\left[\sum_{i=1}^{n} p_i\left(\omega_i t + \phi_i(t) + \frac{\pi}{2}\right)\right], \tag{41}$$

where

$$h_{p_1 p_2 \cdots p_n} = \frac{4}{\pi} \int_0^{\infty} \prod_{i=1}^{n} J_{p_i}(v a_i) \exp\left[-\frac{v^2\sigma^2}{2}\right] \frac{dv}{v} \tag{42}$$

represents the amplitude coefficient of the output signal and intermodulation components.

It is apparent that the determination of all those terms in (41) that will fall in the passband of the filter is quite a formidable task when n is large. Fortunately, Shaft [3] has shown that the dominant output components are the signal and cross-product terms for which $|p_i|_{\max}$ is small (≤ 2). This approximation considerably simplifies (41), since instead of considering all the p_i from $-\infty$ to ∞, it is only necessary to consider them in the range from -2 to 2. The significant filter output components may be

determined by first considering the different frequency combinations in (41), and then selecting the desired ones, i.e., the combinations that will yield frequencies in the passband.

A closed-form solution for the amplitudes of the output components is extremely difficult when n is large. For more than three signals, the amplitudes may be determined either by numerically evaluating the integral of (42), or by using approximations to solve the integral. Shaft [3] used the first approach and has reported extensive computer results for as many as a hundred input signals. Approximate expressions for the amplitudes of the signals and cross products may be obtained by using approximations for the Bessel functions. As an illustration, consider the signal at frequency f_1. The amplitude and phase of the signal is obtained from (41) and (42) by setting $p_1 = \pm 1$ and all the other p_i equal to zero. The result is

$$z_1(t) = h_{10\cdots0} \cos(\omega_1 t + \phi_1(t)), \tag{43}$$

where

$$h_{10\cdots0} = \frac{4}{\pi} \int_0^\infty J_1(va_1) \prod_{i=2}^{n} J_0(va_i) \exp\left[-\frac{v^2\sigma^2}{2}\right] \frac{dv}{v}. \tag{44}$$

An approximate solution to (44) is obtained by using an exponential approximation for the product of Bessel functions, as suggested by Gyi [4]:

$$\prod_{i=2}^{n} J_0(va_i) \approx \exp\left[-\tfrac{1}{2}v^2\sigma_1^2\right], \tag{45}$$

where

$$\sigma_1^2 = \sum_{i=2}^{n} \frac{a_i^2}{2}.$$

The result is then

$$h_{10\cdots0} = 2\sqrt{\frac{\rho}{\pi}} \exp\left[-\frac{\rho}{2}\right] \left[I_0\left(\frac{\rho}{2}\right) + I_1\left(\frac{\rho}{2}\right)\right], \tag{46}$$

where $\rho = a_1^2/[2(\sigma^2 + \sigma_1^2)]$. Bounds on the error for this approximate solution of Gyi have been recently obtained by Campbell [15]. The bound is of the order n^{-1} when the amplitudes of all the signals are approximately equal.

The strongest cross-product terms are those that are produced due to the mixing of any two or three input signals. Approximate expression for the amplitude of these cross products can be obtained from the results for two signals and three signals, respectively. As an example, consider the strongest cross products due to the mixing of the frequencies f_1, f_2, and f_3. They may be obtained from (41) and (42) by considering all the combinations of these three frequencies. The result is

$$g_{\mathrm{IMP}}(t) = -h_{1110\cdots0}[\cos(r+s+t) + \cos(r+s-t) + \cos(r-s+t) + \cos(r-s-t)], \tag{47}$$

where $r = \omega_1 t + \phi_1(t)$, and similarly for s and t. The amplitude coefficient is given by

$$h_{1110\cdots0} = \frac{4}{\pi} \int^\infty J_1(va_1)J_1(va_2)J_1(va_3) \prod_{i=4}^{n} J_0(va_i) \cdot \exp\left[-\frac{v^2\sigma^2}{2}\right] \frac{dv}{v}. \tag{48}$$

Using the exponential approximation to the product of the Bessel functions, the solution of the above integral may be obtained from (21) merely by replacing σ^2 by σ_2^2, where

$$\sigma_2^2 = \sigma^2 + \sum_{i=4}^{n} \frac{a_i^2}{2}.$$

The components that collectively make up the limiter output noise may be obtained by substituting (39) for $s(t)$ in (12), and expanding $\sin[vs(t)]$ and $\cos[vs(t)]$ with the aid of (40). The expansion of $\cos[vs(t)]$ will contain cosine terms instead of sine in (40). The resulting expression then represents the combined output noise produced due to the interaction of the input noise with itself ($n \times n$ terms) and that produced as a result of the interaction of the input signal with noise ($s \times n$ terms). In this expression, the term corresponding to m and all p_i equal to zero represents the direct feedthrough noise ($n \times n$ terms), and all the remaining terms constitute the noise produced by the mixing of the signal with the limiter input noise ($s \times n$ terms). The expression for the direct feedthrough noise ($n \times n$ terms) is given by

$$\eta_{n\times n}(t) = \frac{4}{\pi} \int_0^\infty J_1(vr) \prod_{i=1}^{n} J_0(va_i) \frac{dv}{v} (\cos\omega_0 t + \phi). \tag{49}$$

Davenport [1] has shown, for the case of one signal and noise, that only the $n \times n$ terms contribute significantly to the output noise at low limiter input signal-to-noise ratios. When the number of signals is sufficiently large, so that the amplitude distribution of their sum is approximately Gaussian, the signal-to-noise ratio for any one signal will be small at the limiter input. Consequently, from Davenport's results, it is reasonable to expect that only the direct feedthrough noise will contribute significantly to the limiter output noise.

Again, using the exponential approximation to the product of the Bessel functions (49) can be readily integrated [8] to yield

$$\eta_{n\times n}(t) = \sqrt{\frac{2}{\pi}} \left(\frac{r(t)}{\sigma_3}\right) {}_1F_1\left(\tfrac{1}{2}, 2, -\frac{r^2(t)}{2\sigma_3^2}\right) \cos(\omega_0 t + \phi(t)), \tag{50}$$

where

$$\sigma_3^2 = \sum_{i=1}^{n} \frac{a_i^2}{2}$$

represents the total power of all the input signals.

The power in the feedthrough noise component is given

by

$$N_0 = \frac{8}{\pi^2} E\left[\int_0^\infty J_1(vr) \exp\left[-\tfrac{1}{2}v^2\sigma_3^2\right] \frac{dv}{v}\right]^2$$
$$= \frac{8}{\pi^2} \int_0^\infty \int_0^\infty \exp\left[-\tfrac{1}{2}\sigma_3^2(u^2+v^2)\right]$$
$$\cdot E[J_1(ur)\cdot J_1(vr)] \frac{du}{u}\frac{dv}{v}, \tag{51}$$

where

$$E[J_1(ur)\cdot J_1(vr)] = \frac{1}{\sigma^2}\int_0^\infty J_1(ur)\cdot J_1(vr)\cdot r \cdot \exp\left[-\frac{r^2}{2\sigma^2}\right] dr. \tag{52}$$

This integral is well known as Weber's second exponential integral [14]. Using its solution [14], (51) may be written explicitly as [16]

$$N_0 = \frac{2}{\pi^2}\sum_{n=0}^{\infty} \frac{[\Gamma(n+\frac{1}{2})]^2}{n!\cdot(n+1)!}\left(\frac{\sigma^2}{\sigma^2+\sigma_3^2}\right)^{2n+1}. \tag{53}$$

It is seen that (50) represents the dominant output noise in the time domain. This representation is particularly useful for the evaluation of the system following the bandpass limiter. Equation (50) may also be expressed in the form

$$\eta_{n\times n}(t) = x_1(t)\cos\omega_0 t - y_1(t)\sin\omega_0 t, \tag{54}$$

where $x_1(t)$ and $y_1(t)$, the amplitudes of the "in-phase" and "quadrature" components, are given by

$$x_1(t) = \sqrt{\frac{2}{\pi}}\left(\frac{r(t)}{\sigma_3}\right) {}_1F_1\left(\tfrac{1}{2}, 2, -\frac{r^2(t)}{2\sigma_3^2}\right)\cos\phi(t) \tag{55}$$

$$y_1(t) = \sqrt{\frac{2}{\pi}}\left(\frac{r(t)}{\sigma_3}\right) {}_1F_1\left(\tfrac{1}{2}, 2, -\frac{r^2(t)}{2\sigma_3^2}\right)\sin\phi(t). \tag{56}$$

The functions $x_1(t)$ and $y_1(t)$ are in general non-Gaussian random variables, whose distributions may be obtained from the distribution of r and ϕ.

Conclusions

The effect of ideal bandpass limiting on an arbitrary signal lying in narrowband stationary zero-mean Gaussian noise has been analyzed, and general analytic expressions for the limiter output components have been derived using a time-domain approach. A major advantage of the time-domain method is that it preserves the phases of the signal and intermodulation products, which are destroyed in the characteristic-function method generally used in limiter studies. Expressions for the desired signal and intermodulation product amplitudes have been obtained for the case when the input consists of three angle-modulated sinusoids and noise. The analysis is extended to n angle-modulated sinusoids plus noise, and approximate expressions for the signal, intermodulation product, and noise terms are derived. It has been possible to obtain a time-domain representation for the limiter output noise, which is particularly useful for the evaluation of the system following the bandpass limiter.

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